Annihilating Ideals of Quadratic Forms over Local and Global Fields

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Introduction

It was already known by Witt that the Witt Ring of a field is integral. The same of course holds for the Witt-Grothendieck Ring of isometry classes of quadratic forms over a field.

But only in 1987 did Lewis introduce specific annihilating polynomials. He showed that the polynomials

$$P_n := (X-n)(X-n+2)\cdots(X+n) \in \mathbb{Z}[X], \quad n \in \mathbb{N}_0,$$

annihilate all n-dimensional quadratic forms over an arbitrary field.

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- for the fundamental ideal $I(K)$, K a field with level ≤ 4 (de Wannemacker, 2006).

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2 Annihilating Polynomials

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4. Local Fields
5. Global Fields

Always

- $\mathbb{N}$ does not contain 0. We use $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- We denote by K a field, $\mathrm{char}(K) \neq 2$.
- All quadratic forms are regular (or non-degenerate).

Notation

Let φ, ψ be quadratic forms over K.

- We denote by $\langle a_1, \ldots, a_n \rangle$ the quadratic form over K associated to the diagonal matrix with entries $a_1, \ldots, a_n \in K^*$.

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- $\varphi \perp \psi$ denotes the *orthogonal sum* of φ and ψ.
- $\varphi \otimes \psi$ denotes the *tensor product* of φ and ψ.
- For $m \in \mathbb{N}_0$ we define

$$m \times \varphi \ := \underbrace{\varphi \perp \ldots \perp \varphi}_{m\text{-times}}.$$

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- If φ and ψ are isometric we write $\varphi \cong \psi$.
- $[\varphi]$ denotes the *isometry class* of φ.

The Witt-Grothendieck Ring

The isometry classes of quadratic forms over K form a semi-ring $\widehat{W}^+(K)$ with addition

$$[\varphi] + [\psi] \ := \ [\varphi \perp \psi] \, .$$

and multiplication

$$[\varphi] \cdot [\psi] \ := \ [\varphi \otimes \psi] \, .$$

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By applying the Grothendieck Construction for semi-groups to $\widehat{W}^+(K)$ we obtain the *Witt-Grothendieck Ring* $\widehat{W}(K)$.

The Witt-Grothendieck Ring

The elements of $\widehat{W}(K)$ are formal differences

$$[\varphi] - [\psi]$$

and there exists an up to isometry unique anisotropic form χ over K and a unique $m \in \mathbb{N}$ such that

$$[\varphi] - [\psi] \;=\; [\chi] \pm [m \times \mathbb{H}]\,,$$

where $\mathbb{H} = \langle 1, -1 \rangle$ is the *hyperbolic plane*.

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We extend the notion of *dimension* to a ring homomorphism

$$\dim([\varphi] - [\psi]) \; := \; \dim(\varphi) - \dim(\psi) \; \in \; \mathbb{Z},$$

where $\dim(\varphi)$ denotes the usual dimension of φ.

The Witt Ring

Denote by $\mathcal{H}$ the principal ideal of $\widehat{W}(K)$ generated by $[\mathbb{H}]$.
The *Witt Ring* of K is defined as

$$W(K) := \widehat{W}(K)/_{\mathcal{H}}.$$

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Note

The elements of $W(K)$ classify anisotropic quadratic forms over K.

Annihilating Polynomials

Consider the canonical inclusion

$$\iota : \quad \mathbb{Z} \longrightarrow \widehat{W}(K)$$

defined for $m \in \mathbb{N}$ by

$$m \longmapsto [m \times \langle 1 \rangle] \, .$$

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Definition

A polynomial $P = z_n X^n + \cdots + z_0 \in \mathbb{Z}[X]$ is called *annihilating polynomial* of a quadratic form φ over K, if

$$P([\varphi]) \ := \ z_n [\varphi]^n + \cdots + z_1 [\varphi] + z_n \ = \ 0 \ \in \ \widehat{W}(K).$$

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He furthermore proved that P_n annihilates all quadratic forms of dimension n over an arbitrary field K.

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These polynomials constitute the base for all our following observations.

The Embracing Polynomial

Proposition

For every quadratic form φ over K there exists a unique polynomial $Q_\varphi \in \mathbb{Z}[X]$ which divides all annihilating polynomials of φ and has maximal degree among all polynomials with this properties.

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Idea of Proof.

Use the greatest common divisor and Bézout's identity. □

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Q_φ is called embracing polynomial since $\mathrm{Ann}_\varphi \subset (Q_\varphi)$.

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For every n-dimensional quadratic form φ over K, the factor $X - n \in \mathbb{Z}[X]$ divides Q_φ.

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Sketch of Proof.

Recall that Q_φ is a product of linear factors.

The claim follows since $\dim : \widehat{W}(K) \to \mathbb{Z}$ is a ring homomorphism and $\mathbb{Z}$ is an integral domain. □

Examples

Let φ be a quadratic form over K.

- K **Pythagorean.**
  Then $\widehat{W}(K)$ is torsion free.
  $\Longrightarrow$ We have $Q_\varphi([\varphi]) = 0$ and therefore $\mathrm{Ann}_\varphi = (Q_\varphi)$.

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Formally Real Fields

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For each $A \in X_K$ there exists a signature homomorphism

$$W(K) \longrightarrow \mathbb{Z}$$

defined by

$$\{\langle a \rangle\} \longmapsto \begin{cases} 1 & \text{if } a >_A 0, \\ -1 & \text{otherwise.} \end{cases}$$

This homomorphism will be denoted by sign_A.

The Signature Polynomial

Let φ be a quadratic form of dimension n over K.

Set

$$S_\varphi^{\text{sign}} := \{\text{sign}_A(\{\varphi\}) \mid A \in X_K\} \cup \{n\} \subset \mathbb{Z}.$$

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Proof.

We have $\dim(P_\varphi^{\text{sign}}([\varphi])) = 0$ and $\text{sign}_A(P_\varphi^{\text{sign}}([\varphi])) = 0$ for all $a \in X_K$.

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Since $Q_\varphi([\varphi])$ is torsion and Q_φ is a product of linear factors
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Set $s := \text{sign}(\{\varphi\})$ (there exists only one ordering of $\mathbb{R}$).
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Then a complete and minimal set of generators for $\mathrm{Ann}_\varphi \subset \mathbb{Z}[X]$ is given as follows:

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$$\mathrm{Ann}_\varphi = \begin{cases} (2(X-n), (X-n)^2) & \text{if } \det(\varphi) \text{ is a sum of two squares in } K, \\ (4(X-n), (X-n+2)(X-n)) & \text{otherwise.} \end{cases}$$

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Recall

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- It is well-known that quadratic forms over a local field are classified by dimension, determinant and discriminant.
- The ideal $I(K) \subset W(K)$ consisting of equivalence classes of even dimensional quadratic forms is called *fundamental ideal* of $W(K)$.

Proof of the Proposition

Idea of Proof.

Calculate the Hasse invariants and determinants of

$$2(X-n), \quad (X-n)^2, \quad 4(X-n) \quad \text{resp.} \quad (X-n+2)(X-n)$$

applied to φ.

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Then compare these Hasse invariants (resp. determinants) with the Hasse invariants (resp. determinants) of the hyperbolic forms

$$2n \times \mathbb{H}, \quad 2n^2 \times \mathbb{H}, \quad 4n \times \mathbb{H} \quad \text{resp.} \quad (2n^2+2n) \times \mathbb{H}.$$

□

Classification of Annihilating Ideals over Local Fields

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(a) If $\varphi \cong n \times \langle 1 \rangle$, then $\mathrm{Ann}_\varphi = (X - n)$.

(b) If $\varphi \ncong n \times \langle 1 \rangle$, and

  (i) if $-1 \in (K^*)^2$, then $\mathrm{Ann}_\varphi = \big(2(X-n), (X-n)^2\big)$.

Classification of Annihilating Ideals over Local Fields

In the case that the residue field $\overline{K}$ has characteristic $\neq 2$, we can reformulate the proposition using the valuation v_K of K.

Theorem

Let φ be a quadratic form over K, $n = \dim(\varphi)$.
Assume $\mathrm{char}(\overline{K}) \neq 2$.

(a) If $\varphi \cong n \times \langle 1 \rangle$, then $\mathrm{Ann}_\varphi = (X - n)$.

(b) If $\varphi \not\cong n \times \langle 1 \rangle$, and

  (i) if $-1 \in (K^*)^2$, then $\mathrm{Ann}_\varphi = \big(2(X-n), (X-n)^2\big)$.

  (ii) if $-1 \notin (K^*)^2$, then

$$\mathrm{Ann}_\varphi = \begin{cases} \big(2(X-n), (X-n)^2\big) & \text{if } v_K(\det(\varphi)) \text{ is even,} \\ \big(4(X-n), (X-n+2)(X-n)\big) & \text{if } v_K(\det(\varphi)) \text{ is odd.} \end{cases}$$

Global Fields

In this section K is a global field.

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- For every $\nu \in V$ choose a representative $|\cdot|_\nu$ of the class ν.
- Denote by K_ν the completion of K with respect to $|\cdot|_\nu$.
- We can write V as the disjoint union $V = V_\mathbb{R} \cup V_\mathbb{C} \cup V_{\text{fin}}$ such that

$$K_\nu = \begin{cases} \mathbb{R} & \text{for } \nu \in V_\mathbb{R}, \\ \mathbb{C} & \text{for } \nu \in V_\mathbb{C}, \\ \text{local} & \text{for } \nu \in V_{\text{fin}}. \end{cases}$$

First Observations

Let φ be an n-dimensional quadratic form over K, and let $f \in \mathbb{Z}[X]$.

By the Hasse-Minkowski Theorem f is an annihilating polynomial of φ $\iff$ f is an annihilating polynomial of φ_{K_ν} for all $\nu \in V$.

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Since $\mathrm{Ann}_{\varphi_{K_\nu}} = (X - n)$ for all $\nu \in V_{\mathbb{C}}$, and since $X - n$ divides every annihilating polynomial of φ, we do not have to take into account the completions K_ν for $\nu \in V_{\mathbb{C}}$.

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Proposition

$$\mathrm{Ann}_\varphi = \bigcap_{\nu \in V_\mathbb{R} \cup V_{\mathrm{fin}}} \mathrm{Ann}_{\varphi_{K_\nu}}$$

Orderings of Global Fields

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More specifically: Every signature homomorphism $W(K) \to \mathbb{Z}$ factors uniquely as

$$W(K) \xrightarrow{\varepsilon_\nu} W(\mathbb{R}) \xrightarrow{\text{sign}} \mathbb{Z},$$

where ε_ν is induced by the completion $K \hookrightarrow K_\nu$, and sign is the usual signature homomorphism over $\mathbb{R}$.

The Signature Polynomial revisited

Recall that

$$S_\varphi^{\text{sign}} = \{\text{sign}_A(\{\varphi\}) \mid A \in X_K\} \cup \{n\} \subset \mathbb{Z}$$

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By our previous observations we obtain:

Lemma

The signature polynomial P_φ^{sign} annihilates φ_{K_ν} for all $\nu \in V_{\mathbb{R}}$.

Classification of Annihilating Ideals over Global Fields I

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Classification of Annihilating Ideals over Global Fields II

Theorem

Let φ be a quadratic form over K, $n = \dim(\varphi)$.

(b) If $\varphi \not\cong n \times \langle 1 \rangle$, and

(ii) if $|S_\varphi^{\text{sign}}| = 2$ with $S_\varphi^{\text{sign}} = \{s, n\}$, then

$$\mathrm{Ann}_\varphi = \begin{cases} \big(2(X-s)(X-n), (X-s)(X-n)^2\big) & \text{if } s \equiv n \pmod 4 \text{ and } \det(\varphi_{K_\nu}) \text{ is not a sum of two squares in } K_\nu \text{ for some } \nu \in V_{\text{fin}}, \\ \big((X-s)(X-n)\big) & \text{otherwise.} \end{cases}$$

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(iii) if $|S_\varphi^{\text{sign}}| \geq 3$, then $\mathrm{Ann}_\varphi = (P_\varphi^{\text{sign}})$.

Proof

Sketch of Proof.

The claims (a) and (b).(iii) follow directly from our previous observations.

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The claims (a) and (b).(iii) follow directly from our previous observations.

For the proof of (b).(i) one shows that one does not have to take into account the real completions K_ν for $\nu \in V_{\mathbb{R}}$.

The proof of (b).(ii) involves some Hasse invariant calculations and arguments analogous to those used to prove the classification of annihilating ideals over local fields. □

The End.